Input to State Stabilizing Control Lyapunov Functions for Robust Bipedal Robotic Locomotion

Shishir Kolathaya, Jacob Reher, Ayonga Hereid and Aaron D. Ames

Abstract— **This paper analyzes the input to state stability properties of controllers which stabilize hybrid periodic orbits. Systems that are input to state stable tend to be robust to modeling and sensing uncertainties. The main contribution of this paper is in the construction of a class of control Lyapunov functions that do not just stabilize, but also input to state stabilize a given hybrid system. Bipedal robotic walking, which can be naturally modeled as a hybrid system, is analyzed under this class of controllers. Specifically, we will select a class of controllers called *rapidly exponentially stabilizing control Lyapunov functions* that stabilize bipedal robotic walking; typically modeled as hybrid periodic orbits. We will show with simulation results that there exists a subset of this class of control Lyapunov functions that input to state stabilizes the given stable periodic orbit; called the input to state stabilizing control Lyapunov functions.**

I. Introduction

Controllers which operate on the states and model of a system are highly sensitive to imperfections in real world implementations. This mismatch is especially relevant in the field of bipedal robotics and can affect controllers adversely through imperfect sensing, inaccurate parameter estimation, input saturations and unmodeled disturbances, to name a few. The notion of *input to state stability* (ISS) [16] captures this uncertainty in a way in which the deviation from the desired output is a function of the deviation from the stabilizing control input. Practical difficulties in the realization of nonlinear feedback controllers on robotic systems places heavy constraints on the ability to increase control gains, and thus to improve convergence rates and tracking errors.

Control Lyapunov Functions (CLF), popularized by Artstein and Sontag [14] during the $1980's$, enable the use of dynamic programming approaches to obtain optimal control inputs in real-time controllers [6]. Some examples include [10], [4], [8]. The translation of this approach to hybrid systems, especially bipedal robotic systems with underactuation and discrete jumps (impacts), brings with itself a larger challenge. For complex systems such as these, investigating input to state stability (ISS), i.e., studying output perturbations for all kinds of input perturbations seems like an unavoidable task. Indeed, input to state stability of hybrid systems has been studied extensively in literature. Some of the problems addressed are finding a common Lyapunov function [22], [7] and stability under fast switching [11]. There are also interesting properties of several hybrid systems with discrete events that can stabilize continuous unstable dynamics [20], [23].

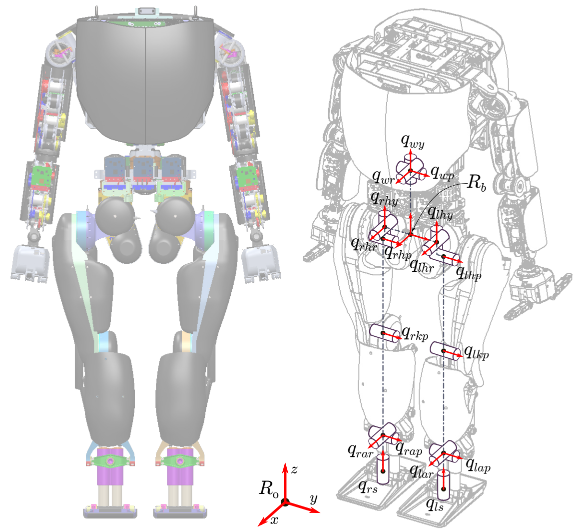

Fig. 1: DURUS robot designed by SRI International.

Our quest for desirable stabilization properties under uncertainty lends itself naturally to CLFs. The focal point of this paper is to study and analyze the ISS properties of CLFs for stabilizing nonlinear hybrid systems with affine control inputs; specifically applying to bipedal walking robots. Through the use of constructions given by Sontag [13], we show that it is indeed possible to find a subset of input to state stabilizing CLFs from a given set of CLFs. The core advantage is the increase in the number of choices from just one to infinitely many, a necessity for optimal control approaches. We will consider stable walking gaits, i.e., stable hybrid periodic orbits and obtain the ISS properties of these orbits under an ISS controller. Comparisons are also made between two specific controllers in simulation: Feedback Linearization and its ISS equivalent, for the humanoid robot DURUS (Fig. 1), and show robustness to pushing, uncertain terrain height, and model perturbations.

The paper is structured as follows: Section II contains a brief preliminary on input to state stability. Section III introduces a set of asymptotically stabilizing control Lyapunov functions (AS-CLFs), exponentially stabilizing control Lyapunov functions (ES-CLFs) and finally the set of rapidly

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exponentially stabilizing control Lyapunov functions (RES-CLFs). This section will also define the set of input to state stabilizing control Lyapunov functions (ISS-CLFs) and show how to obtain the subset of ISS-CLFs from the given set of CLFs. RES-CLFs are important for the exponential stabilization of hybrid periodic orbits (see [4]). Section IV will introduce the definition of hybrid systems and the notion of hybrid zero dynamics. Section V introduces the main theorem and Section VI shows the simulation results and comparisons between a standard stabilizing controller and its ISS equivalent.

II. Preliminaries on Input to State Stability

This section will introduce basic definitions and results related to input to state stability (ISS); for a detailed survey on ISS see [16]. Most of the content in this section is based on [13], [15], [12], [18], [17]. A general nonlinear system with outputs is represented in the following fashion:

$$\dot{x} = f(x,u), \tag{1}$$

with x taking values in Euclidean space $\mathbb{R}^n$, the input $u \in \mathbb{R}^m$, the output $y \in \mathbb{R}^k$ for some positive integers n,m,k. The mapping $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is considered Lipschitz and $f(0,0) = 0$. We can define a set of outputs $y : \mathbb{R}^n \to \mathbb{R}^k$ which is continuous with $y(0) = 0$. We use a feedback control law

$$u = k(x), k(0) = 0, \tag{2}$$

that makes the closed loop system

$$\dot{x} = f(x,k(x)) =: f_{cl}(x), \tag{3}$$

globally asymptotically stable about $x = 0$. We say that a controller, $k(x)$, is stabilizing if it makes the closed loop system (3) globally asymptotically stable. Mathematically, the notion of input/output stability arises from the need to find a feedback (2) with the property that the new control system

$$\dot{x} = f(x,k(x)+d), \tag{4}$$

be *input to state stable*. For partial observations of interest, we use the notion of input d to output y stabilization.

It is a well known fact that the feedback control law $k(x)$ which achieves state-space stabilization does not necessarily produce input/output stabilization. It is specifically the classes of systems satisfying this property which are of interest to us. We will state some basic definitions of stability here: $\mathbb{R}^n$ denotes n dimensional Euclidean space, the field $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_{\geq 0}$ denotes nonnegative real numbers. $\mathbb{N}$ is the set of nonnegative integers. The vectors $y_1 \in \mathbb{R}^{k_1}, y_2 \in \mathbb{R}^{k_2}$, when combined together into one vector are denoted by $(y_1,y_2) = [y_1^T, y_2^T]^T \in \mathbb{R}^{k_1+k_2}$. $|x|$ is the Euclidean vector norm. The infinity norm on the vector is denoted by $\|d\|_\infty := \sup_{t\geq 0}\{|d(t)|\}$. The matrix norm of a matrix is denoted by $\|P\|$, which is its maximum eigen value.

Class $\mathscr{K}, \mathscr{K}_\infty$ and $\mathscr{K}\mathscr{L}$ functions. A class $\mathscr{K}$ function is a function $\alpha : [0,a) \to \mathbb{R}_{\geq 0}$ which is continuous, strictly increasing and satisfies $\alpha(0) = 0$. A class $\mathscr{K}_\infty$ function is a function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which is continuous, strictly increasing, proper, and satisfies $\alpha(0) = 0$, and a class $\mathscr{K}\mathscr{L}$ function is a function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\beta(r,t) \in \mathscr{K}_\infty$ for each t and $\beta(r,t) \to 0$ as $t \to \infty$.

We can now define ISS to consider the entire dynamics of (4). It is important to note that the input considered for ISS is the disturbance d. Therefore, all ISS and related definitions are w.r.t. d.

Definition 1: *The system (4) is input to state stable (ISS) if there exists $\beta \in \mathscr{K}\mathscr{L}$, and $\iota \in \mathscr{K}_\infty$ such that*

$$|x(t,x_0,d)| \leq \beta(|x_0|,t) + \iota(\|d\|_\infty), \qquad \forall x_0, \forall t \geq 0, \tag{5}$$

and (4) is considered locally ISS, if the inequality (5) is valid for an open ball of radius r, $x_0 \in \mathbb{B}_r(0)$.

Definition 2: *The system (4) is exponential input to state stable (e-ISS) if there exists $\beta \in \mathscr{K}\mathscr{L}$, $\iota \in \mathscr{K}_\infty$ and a positive constant $\lambda > 0$ such that*

$$|x(t,x_0,d)| \leq \beta(|x_0|,t)e^{-\lambda t} + \iota(\|d\|_\infty), \qquad \forall x_0, \forall t \geq 0, \tag{6}$$

and (4) is considered locally e-ISS, if the inequality (6) is valid for an open ball of radius r, $x_0 \in \mathbb{B}_r(0)$.

Definition 3: *The system (4) is said to hold the asymptotic gain (AG) property if there exists $\iota \in \mathscr{K}_\infty$ such that*

$$\overline{\lim}_{t\to\infty}|x(t,x_0,d)| \leq \iota(\|d\|_\infty), \qquad \forall x_0, d. \tag{7}$$

Definition 4: *The system (4) is said to be zero stable (ZS) for a zero input $d = 0$, if there exists $\beta \in \mathscr{K}\mathscr{L}$ such that*

$$|x(t,x_0,0)| \leq \beta(|x_0|,t), \qquad \forall x_0, \forall t \geq 0. \tag{8}$$

Fig. 2 depicts the ZS and AG property.

Lemma 1: *The system (4) is ISS if and only if it is ZS and AG.*

Input to State Stable Lyapunov functions. A direct consequence of using ISS concepts is the construction of input to state stable Lyapunov functions.

Definition 5: *An ISS-Lyapunov function for (4) is a continuously differentiable positive definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for which there exist functions $\underline{\alpha}, \bar{\alpha}, \alpha, \iota, \in \mathscr{K}_\infty$ such that*

$$\begin{aligned} &\underline{\alpha}(x) \leq V(x) \leq \bar{\alpha}(x), \\ &\dot{V}(x,d) \leq -\alpha(|x|) + \iota(\|d\|_\infty), \qquad \forall x, d. \end{aligned} \tag{9}$$

Input to State Stabilty of Affine Control Systems. We can consider affine control systems of the form

$$\dot{x} = f(x) + g(x)u, \tag{10}$$

where $g(x) \in \mathbb{R}^{n\times m}$, is a Lipschitz function of x. Similar to (1), it is assumed that $f(0) = 0$, and both f,g are Lipschitz in x. The main result of [13] was that systems of the type (10) can be made input to state stable. In other words, there exists a Lipschitz continuous map $k : \mathbb{R}^n \to \mathbb{R}^m$, with $k(0) = 0$, and the control law $k(x)+d$, such that the new system

$$\dot{x} = f(x) + g(x)k(x) + g(x)d \tag{11}$$

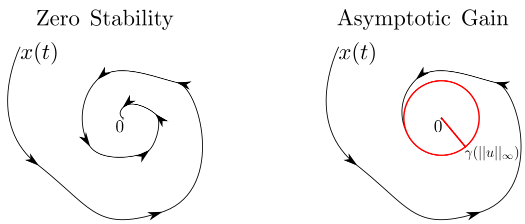

Fig. 2: ZS is achieved for a zero input, and AG is achieved for a bounded input. We use these two important properties to prove input to state stability.

is ISS.

We can now also define the notion of ISS stabilizability which is a powerful tool to obtain controllers that satisfy the ISS conditions [13].

Definition 6: *The system (10) is smoothly stabilizable, if there is a smooth map $k : \mathbb{R}^n \to \mathbb{R}^m$ with $k(0) = 0$ such that the system (10) is GAS. It is smoothly input to state stabilizable if there is a k so that the system (11) becomes ISS.*

Based on this definition, we have this powerful Lemma that is taken from Theorem 1 of [13] and restated here.

Lemma 2: *For systems of the type (10) smooth stabilizability implies smooth input to state stabilizability.*

It is important to note that the smoothness property can be relaxed to just Lipschitz continuity property without violating Lemma 2.

III. Input to State Stabilizing Control Lyapunov Functions

The goal of this section is to generalize and define the set of stabilizing controllers (i.e., not just one $k(x)$) via control Lyapunov functions (CLFs) that yield ISS. Specifically, in the context of ISS, we will derive a sub-class of control Lyapunov functions that *input to state stabilize* the system (1). CLFs are obtained for the control input u, and the ISS conditions are satisfied for the disturbance input d. We would like to call these CLFs *input to state stabilizing control Lyapunov functions (ISS-CLFs)*. Towards the end of this section, we will derive a subclass for *rapidly exponentially stabilizing control Lyapunov functions (RES-CLF)* that are important leading into the next section (for hybrid systems).

Definition 7: *For the system (1), a continuously differentiable function $V : \mathbb{R}^{n\times m} \to \mathbb{R}_{\geq 0}$ is an asymptotically stabilizing control Lyapunov function (AS-CLF), if there exists a set of admissible controls $\mathbb{U}$, and $\underline{\alpha}, \bar{\alpha}, \alpha \in \mathcal{K}_\infty$ such that*

$$\begin{aligned} &\underline{\alpha}(|x|) \leq V(x) \leq \bar{\alpha}(|x|) \\ &\inf_{u\in\mathbb{U}} [L_f V(x,u) + \alpha(|x|)] \leq 0. \end{aligned} \tag{12}$$

We are interested in affine systems of the form (10) which represents a large class of systems like robotic systems. So we can similarly define CLFs for such systems by replacing the Lie derivative $L_f V$ with $L_f V(x) + L_g V(x)u$.

We will define the *exponentially stablilizing control Lyapunov function* (ES-CLF), for affine systems here.

Definition 8: *For the system (10), a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an exponentially stablilizing control Lyapunov function (ES-CLF) if there exist positive constants $\underline{c}, \bar{c}, c > 0$ such that for all x*

$$\begin{aligned} &\underline{c}\|x\|^2 \leq V(x) \leq \bar{c}\|x\|^2 \\ &\inf_{u\in\mathbb{U}} [L_f V(x) + L_g V(x)u + cV(x)] \leq 0, \end{aligned} \tag{13}$$

L_f, L_g are the Lie derivatives. We can accordingly define a set of controllers which render exponential convergence of the states to 0

$$\mathbf{K}(x) = \{u \in \mathbb{U} : L_f V(x) + L_g V(x)u + cV(x) \leq 0\}, \tag{14}$$

which has the control values that result in $\dot{V} \leq -cV$.

Input to State Stabilizing Control Lyapunov Functions. We define here, a sub-class of CLFs, that are input to state stable. In other words, we define *input to state stabilizing control Lyapunov functions (ISS-CLFs)*. We will use ISS-Lyapunov functions that are defined in Section II.

Definition 9: *For the system (1), an asymptotically stabilizing CLF, $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (Definition 7), is an input to state stable stabilizing control Lyapunov function (ISS-CLF), if it is satisfies the conditions of an ISS-Lyapunov function. In other words, if there is $\iota \in \mathcal{K}_\infty$ such that*

$$\begin{aligned} &\dot{V}(x,u,d) \leq -\alpha(|x|) + \iota(\|d\|_\infty) \\ &\text{for } u = \arg\{\inf_{u\in\mathbb{U}} [L_f V(x) + L_g V(x)u + \alpha(|x|)] \leq 0\} \end{aligned} \tag{15}$$

Motivated by constructions of input to state stabilizable controllers developed by Sontag, specifically, equations (23) and (32) in [13], we can construct ISS-CLFs in the following manner. Parts of these are also derived from Artstein's theorem [5], [14]. Considering the stabilizing controller $k(x)$ that resulted in the closed loop system (3) again, we have the Lie derivative w.r.t. the closed loop vector field f_{cl} as $L_{f_{cl}} V(x) = \frac{\partial V}{\partial x} f_{cl}(x)$. It was shown in [13] that the controller

$$u = k(x) + \frac{1}{2m} L_{f_{cl}} V(x) L_g V(x)^T, \tag{16}$$

input to state stabilizes the system (1). We can derive a controller like the following which also renders the system (4) ISS:

$$u = k(x) - \frac{1}{\bar{\varepsilon}} L_g V(x)^T, \tag{17}$$

for some $\bar{\varepsilon} > 0$. Based on this controller, we have the following Lemma which defines a new set of CLFs that input to state stabilizes the system (1).

Lemma 3: *The continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined for $\underline{\alpha}, \bar{\alpha}, \alpha \in \mathcal{K}_\infty$ as*

$$\begin{aligned} &\underline{\alpha}(|x|) \leq V(x) \leq \bar{\alpha}(|x|) \\ &\inf_{u\in\mathbb{U}} [L_f V(x) + L_g V(x)u + \alpha(|x|) + \frac{1}{\bar{\varepsilon}} L_g V(x) L_g V(x)^T] \leq 0, \end{aligned} \tag{18}$$

is an ISS-CLF $\forall\ \bar{\varepsilon} > 0$.

Proof: We have the following expression after substituting the controller (18) in the derivative of the Lyapunov function.

$$\begin{aligned}\dot{V}(x,u,d) &= L_fV(x) + L_gV(x)u + L_gV(x)d \qquad (19)\\ &\leq -\alpha(|x|) - \frac{1}{\bar{\varepsilon}}L_gV(x)L_gV(x)^T + L_gV(x)d.\end{aligned}$$

Since $L_gV(x) \in \mathbb{R}^{1\times m}$, $\|L_gV\|^2 \geq 0$. So we have the following inequality after adding and subracting $\bar{\varepsilon}\frac{\|d\|_\infty^2}{4}$

$$\begin{aligned}\dot{V}(x,u,d) &\leq -\alpha(|x|) - \frac{1}{\bar{\varepsilon}}\|L_gV(x)\|^2 + \|L_gV(x)\|\|d\|_\infty\\ &\quad -\bar{\varepsilon}\frac{\|d\|_\infty^2}{4} + \bar{\varepsilon}\frac{\|d\|_\infty^2}{4}\\ &\leq -\alpha(|x|) - \left(\frac{1}{\sqrt{\bar{\varepsilon}}}\|L_gV(x)\| - \sqrt{\bar{\varepsilon}}\frac{\|d\|_\infty}{2}\right)^2 + \bar{\varepsilon}\frac{\|d\|_\infty^2}{4}\\ &\leq -\alpha(|x|) + \bar{\varepsilon}\frac{\|d\|_\infty^2}{4}, \qquad (20)\end{aligned}$$

which is in the form given by (9). It can be observed that an excellent way to drive the ultimate bound to a very small value is by decreasing $\bar{\varepsilon}$. ■

If we pick an exponentially stabilizing control Lyapunov function (ES-CLF), (13), we can modify (20) that results in exponential input to state stability (e-ISS) w.r.t. d.

Lemma 4: *The continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined for $\underline{c}, \bar{c}, c > 0$ as*

$$\underline{c}\|x\|^2 \leq V(x) \leq \bar{c}\|x\|^2 \qquad (21)$$

$$\inf_{u\in\mathbb{U}}[L_fV(x) + L_gV(x)u + cV(x) + \frac{1}{\bar{\varepsilon}}L_gV(x)L_gV(x)^T] \leq 0,$$

is an e-ISS-CLF $\forall\ \bar{\varepsilon} > 0$.

Proof of this is omitted since it is straightforward from (20), where $\alpha(|x|)$ needs to be simply replaced with $cV(x)$. Motivated by Lemma 3 we can create a subclass of ES-CLFs that are e-ISS

$$\begin{aligned}\mathbf{K}_{\bar{\varepsilon}}(x) = \{u \in \mathbb{U} :& L_fV(x) + L_gV(x)u + cV(x)\ldots\\ &+ \frac{1}{\bar{\varepsilon}}L_gV(x)L_gV(x)^T \leq 0\}. \qquad (22)\end{aligned}$$

Since $\bar{\varepsilon}, L_gV(x)L_gV(x)^T$ are both ≥ 0, it can be verified that $\mathbf{K}_{\bar{\varepsilon}} \subseteq \mathbf{K}$ (the set obtained from (14)).

Rapidly Exponentially Stabilizing Control Lyapunov Functions. If we need stronger bounds of convergences (especially used for hybrid systems like bipedal robots; more on in this is discussed in Section IV), a *rapidly exponentially stabilizing control Lyapunov function (RES-CLF)* is constructed that stabilizes the output dynamics at a rapidly exponential rate (see [4] for more details) through a user defined $\varepsilon > 0$.

Definition 10: The family of continuously differentiable functions $V_\varepsilon : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a *rapidly exponentially stabilizing control Lyapunov function (RES-CLF)* if there exist positive constants $c_1, c_2, c_3 > 0$ such that for all $0 < \varepsilon < 1$, x,

$$c_1\|x\|^2 \leq V_\varepsilon(x) \leq \frac{c_2}{\varepsilon^2}\|x\|^2, \qquad (23)$$

$$\inf_{u\in\mathbb{U}}[L_fV_\varepsilon(x) + L_gV_\varepsilon(x)u + \frac{c_3}{\varepsilon}V_\varepsilon(x)] \leq 0.$$

Therefore, we can define a class of controllers $\mathbf{K}_\varepsilon$:

$$\mathbf{K}_\varepsilon(x) = \{u \in \mathbb{U} : L_fV_\varepsilon(x) + L_gV_\varepsilon(x)u + \frac{\gamma}{\varepsilon}V_\varepsilon(x) \leq 0\}, \quad (24)$$

which yields the set of control values that satisfies the desired convergence rate.

Similar to Lemma 4 we pick a subclass of RES-CLFs called *rapidly exponentially input to state stabilizing control Lyapunov functions (Re-ISS-CLF)* which is given in the lemma below.

Lemma 5: *The continuously differentiable function $V_\varepsilon : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined for $c_1, c_2, c_3 > 0$ as*

$$c_1\|x\|^2 \leq V_\varepsilon(x) \leq \frac{c_2}{\varepsilon^2}\|x\|^2, \qquad (25)$$

$$\inf_{u\in\mathbb{U}}[L_fV_\varepsilon(x) + L_gV_\varepsilon(x)u + \frac{c_3}{\varepsilon}V_\varepsilon(x) + \frac{1}{\bar{\varepsilon}}L_gV_\varepsilon(x)L_gV_\varepsilon(x)^T] \leq 0,$$

is an Re-ISS-CLF $\forall\ 0 < \varepsilon < 1,\ \bar{\varepsilon} > 0$.

Motivated by Lemma 3 we can create a subclass of RES-CLFs that are Re-ISS

$$\begin{aligned}\mathbf{K}_{\varepsilon,\bar{\varepsilon}}(x) = \{u \in \mathbb{U} : L_fV(x) + L_gV(x)u + &\frac{c_3}{\varepsilon}V_\varepsilon(x) \qquad (26)\\ &+ \frac{1}{\bar{\varepsilon}}L_gV_\varepsilon(x)L_gV_\varepsilon(x)^T \leq 0\}.\end{aligned}$$

Similar to (22), $\bar{\varepsilon}, L_gV(x)L_gV(x)^T$ are both ≥ 0, it can be verified that $\mathbf{K}_{\varepsilon,\bar{\varepsilon}} \subseteq \mathbf{K}_\varepsilon$ (the set obtained from (24)). In fact, $\mathbf{K}_{\varepsilon,\bar{\varepsilon}} \subseteq \mathbf{K}_{\bar{\varepsilon}} \subseteq \mathbf{K}$, and $\mathbf{K}_\varepsilon \subseteq \mathbf{K}$.

To summarize, for affine systems of the form (10), we showed that we can create a set of ISS controllers for the three types of classes: AS-CLFs, ES-CLFs, RES-CLFs. We also defined two subclasses mathematically, e-ISS-CLF (22), and Re-ISS-CLF (26) which both yield e-ISS. The purpose of Re-ISS-CLF will be more clear in the context of hybrid systems in Section IV.

Extension of Artstein's Theorem We can make use of Artstein's theorem [5] which states that existence of a smooth control Lyapunov function implies smooth stabilizability, which can be extended to include ISS-CLFs by using Lemma 2.

Lemma 6: *For systems of the form* (10)*, existence of a smooth stabilizing control Lyapunov function implies the existence of a smooth input to state stabilizing control Lyapunov function.*

This, of course, is valid for non-smooth Lipschitz continuous control laws, and can also be extended to AS-CLFs, ES-CLFs and RES-CLFs and conclude that the corresponsing ISS versions of these CLFs can be easily computed. Lemma 6 can be used to construct input to state stabilizing controllers, given a CLF. While constructing these robust controllers are possible, major challenges lie in obtaining a Lyapunov function for the given system. The most popular approach used is via feedback linearization. More on this is explained in Section VI.

IV. Hybrid Systems

In this section, we will discuss a general hybrid model; generally used for a bipedal robots. Informally, a hybrid

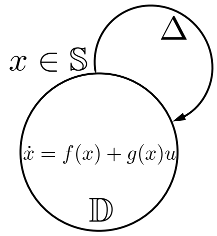

Fig. 3: Figure showing a simple hybrid system.

system is an alternating sequence of continuous and discrete events.

Definition 11: A hybrid control system is defined to be the tuple:

$$\mathscr{HC} = (\Gamma, \mathbb{D}, \mathbb{U}, \mathbb{S}, \Delta, \mathbb{FG}), \tag{27}$$

where each element in the set $\mathscr{HC}$ is described below: The directed graph: $\Gamma = (\mathbb{V}, \mathbb{E})$ consisting of vertices and edges:

$$\mathbb{V} = \{v_1, v_2, \dots\},\ \mathbb{E} = \{e_1, e_2, \dots\}, \tag{28}$$

$\mathbb{D} = \{\mathbb{D}_{v_1}, \mathbb{D}_{v_2}, \dots\}$ is the set of domains. Each domain $\mathbb{D}_v$, with $v \in \mathbb{V}$ can be defined as the set of feasible states. There could be kinematic and dynamic contraints that could limit the statespace. $\mathbb{S} = \{\mathbb{S}_{e_1}, \mathbb{S}_{e_2}, \dots\}$ is the set of switching surfaces or guards, with each guard $\mathbb{S}_e$, representing the surface where the switch over to the next domain happens. $\mathbb{U} = \{\mathbb{U}_{v_1}, \mathbb{U}_{v_2}, \dots\}$ is the set of admissible control inputs. If a feedback control law $u_v = k(x) \in \mathbb{U}_v \subseteq \mathbb{R}^m$ is implemented, the hybrid control system $\mathscr{HC}$ reduces to a hybrid system $\mathscr{H}$ with the omission of $\mathbb{U}$. $\Delta = \{\Delta_{e_1}, \Delta_{e_2}, \dots\}$ is the set of switching functions or reset maps from one domain to the next domain. Each reset map $\Delta_e : \mathbb{S}_{v_{\text{source}}} \to \mathbb{D}_{v_{\text{target}}}$, with $e = \{v_{\text{source}} \to v_{\text{target}}\} \in \mathbb{E}$ is computed at the end of every continuous event. It is assumed that the reset maps are Lipschitz continuous in x. $\mathbb{FG} = \{(f_{v_1}, g_{v_1}), (f_{v_2}, g_{v_2}), \dots\}$ provides the set of vector fields given by the equation:

$$\dot{x} = f_v(x) + g_v(x)u,\ \ u \in \mathbb{U}_v. \tag{29}$$

f_v, g_v are both assumed to be Lipschitz continuous in x. If the hybrid system is modelling a mechanical (robotic) system, then the vector of its configuration q and velocities $\dot{q}$ can be put together: $x = (q, \dot{q})$ (see (30)).

A system with a single domain and single resetmap is called a simple hybrid control system (see Fig. 3).

A wide variety of hybrid systems can be defined in the form of (27) like mechanical systems, network control systems, switching power systems, embedded systems and so on. This representation is derived from category theory used in [3]. Since we are specifically interested in affine systems, we have considered the vector fields of the form (29) (although it is easily extensible to non affine systems).

Input to State Stability of Hybrid Systems. Input to state stability for hybrid systems is defined similar to continuous systems, with the norm on the inputs being the maximum value of the suprema of the inputs in during each continuous event. $\|d\|_{\mathbb{V}} = \max_{v \in \mathbb{V}} \|d_v\|_\infty$ Due to space contraints, we will not define explicitly, but all the definitions are directly derived from Definition 1 to 5. Moreover, since we are interested in the stability properties of hybrid periodic orbits, will use the Poincaré map analysis derived from [4].

Trajectory Tracking Control. We now can describe the trajectory tracking controllers for DURUS (Fig. 1) here. We will consider a general EOM of the form

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B(q)u + \mathscr{J}_v^T F_v, \tag{30}$$

where $\mathscr{J}_v$ Jacobian of the holonomic constraints of DURUS. Accordingly, we have the dynamics represented in terms of the state $x = (q, \dot{q})$.

$$\dot{x} = f_v(x) + g_v(x)u \tag{31}$$
$$f_v(x) = \begin{bmatrix} \dot{q} \\ D^{-1}(q)(-C(q,\dot{q})\dot{q} - G(q) + \mathscr{J}_v^T F_v) \end{bmatrix}, g_v(x) = \begin{bmatrix} 0 \\ D^{-1}(q)B \end{bmatrix}.$$

Outputs. The goal is to derive controllers that realize a gait in the bipedal robot. The problem is setup such that the objective is to drive the robot states to a reference periodic orbit by a tracking control law. We have the set of actual outputs of the robot as $y^a : T_q\mathbb{Q} \to \mathbb{R}^k$, and the desired outputs as $y^d : \mathbb{R}^+ \to \mathbb{R}^k$. y^d is modulated by a phase variable $\tau : \mathbb{Q} \to \mathbb{R}^+$. By adapting a IO linearizing controller, we can drive the relative degree one outputs (velocity outputs)

$$y_{1,v}(q,\dot{q}) = y^a_{1,v}(q,\dot{q}) - y^d_{1,v}(\alpha_v), \tag{32}$$

and relative two outputs (pose outputs)

$$y_{2,v}(q) = y^a_{2,v}(q) - y^d_{2,v}(\tau, \alpha_v), \tag{33}$$

to zero, with v denoting the domain, α denoting the parameters of the desired trajectory. These outputs are generally called *virtual constraints* [21]. The phase variable, τ, is used for modulating only the relative degree two outputs. Walking gaits, viewed as a set of desired periodic trajectories, are modulated as functions of a phase variable to eliminate the dependence on time [19]. The IO linearizing controller that drives the purely state dependent outputs $y_{1,v} \to 0$, $y_{2,v} \to 0$ is given by:

$$u_{\text{IO}} = \begin{bmatrix} L_g y_{1,v} \\ L_g L_f y_{2,v} \end{bmatrix}^{-1} \left(- \begin{bmatrix} L_f y_{1,v} \\ L_f^2 y_{2,v} \end{bmatrix} + \mu \right), \tag{34}$$

where μ denotes the auxiliary input applied after the feedback linearization. Note that the IO controller is one member of the class of CLFs. Denote $\eta_v = (y_{1,v}, y_{2,v}, \dot{y}_{2,v})$. If the system has outputs with more relative degrees of freedom, then η can be accordingly modified. Applying the controller (34) results in the following output dynamics

$$\dot{\eta}_v = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1_{k_2 \times k_2} \\ 0 & 0 & 0 \end{bmatrix}}_{F} \eta_v + \underbrace{\begin{bmatrix} 1_{k_1 \times k_1} & 0 \\ 0 & 0 \\ 0 & 1_{k_2 \times k_2} \end{bmatrix}}_{G} \mu. \tag{35}$$

k_1 is the size of the velocity outputs $y_{1,v}$, and k_2 is the size of the relative degree two outputs $y_{2,v}$. The dimension of the outputs $k_1 + k_2 = k$ is typically equal to the number of actuators m.

Hybrid Zero Dynamics. When the control objective is met such that $\eta_v = 0$ for all time then the system is said to be operating on the *zero dynamics surface* [2] represented by the coordinates $z_v \in \mathbb{R}^{2n-k_1-2k_2}$. Further, by relaxing the zeroing of the velocity output $y_{1,v}$, we can realize *partial zero dynamics*

$$\mathbb{PZ}_v = \{(q,\dot{q}) \in \mathbb{D}_v | y_{2,v} = 0, L_f y_{2,v} = 0\}. \quad (36)$$

The humanoid robot, DURUS (Fig. 1), has feet and employs ankle actuation to propel the hip forward during the continuous dynamics. The relaxation assumption is implemented on the hip velocity, resulting in *partial zero dynamics*. The controller drives the outputs to partial zero dynamic surface only in the continuous dynamics. Therefore, for a hybrid control system $\mathscr{HC}$, *partial hybrid zero dynamics* can be guaranteed if and only if the discrete maps $\{\Delta_e\}_{e\in\mathbb{E}}$ are invariant of the partial (or full) zero dynamics in each domain.

$$\Delta_e(\mathbb{PZ}_{v_{\text{source}}} \cap \mathbb{S}_{v_{\text{source}}}) \subset \mathbb{PZ}_{v_{\text{target}}}, \; e = \{v_{\text{source}} \to v_{\text{target}}\} \in \mathbb{E}. \quad (37)$$

The best way to ensure hybrid invariance under a discrete transition is by a careful selection of the desired trajectories (desired gait) via the parameterization: α_v, which are chosen by using a direct collocation based walking gait optimization problem which is explained in [9]. The outputs η_v are normally called the transverse coordinates, and z_v are called the zero coordinates, and they are related to x via the diffeomorphism $(\eta_v, z_v) = \Phi_v(x)$ and

$$\eta_{2,v} := \begin{bmatrix} y_{2,v} \\ L_f y_{2,v} \end{bmatrix} =: \Phi_v^{y_2}(x), \zeta_v := \begin{bmatrix} y_{1,v} \\ z \end{bmatrix} =: \Phi_v^z(x). \quad (38)$$

V. Formal Results of Stability

In this section, we will investigate the stability properties of hybrid systems for controllers of the form (24) first and then analyze ISS properties of controllers of the form (26). By viewing walking as periodic orbits that are hybrid in nature, we check for conditions that result in attractive and forward invariant periodic orbits. Applying the RES-CLF (24) on the hybrid control system $\mathscr{HC}$ (27) yields the hybrid system

$$\mathscr{H}^\varepsilon = (\Gamma, \mathbb{D}, \mathbb{S}, \Delta, \mathbb{F}^\varepsilon), \quad (39)$$

where the only difference is the set of vector fields $\mathbb{F}^\varepsilon = \{f^\varepsilon_{v_1}, f^\varepsilon_{v_2}, \dots\}$, which are obtained after the substitution of (24). Similarly, applying Re-ISS-CLF (26) instead of (24) results in the hybrid system $\mathscr{H}^{\varepsilon,\bar{\varepsilon}}$.

Since the same controller (24) drives the outputs $\eta_{2,v} \to 0$, implying (partial) zero dynamics. Hybrid invariance (37) and convergence of outputs $\eta_{2,v} \to 0$ results in a reduced order hybrid system consisting of only the coordinates ζ_v (from (38)).

$$\mathscr{H}|_z = (\Gamma, \mathbb{D}|_z, \mathbb{S}|_z, \Delta|_z, \mathbb{F}|_z). \quad (40)$$

The dynamics observed in this reduced order hybrid system is called the partial hybrid zero dynamics.

Periodic Orbits and Poincaré maps. The solution $x(t,x_0)$ is periodic if there is a period T^* such that $x(t+T^*, x_0) = x(t,x_0)$. By transforming x into the coordinates (η_v, z_v), we can define a hybrid flow for the transformed coordinates (transverse and zero) $\varphi_t(\eta_v, z_v)$. Therefore, when the solution passes through two domains with the transition at t_1, we have $\varphi_{t_1+t_2} = \varphi_{t_2} \circ \Delta_{e_1} \circ \varphi_{t_1}(\eta_v, z_v)$. We can also define a periodic flow $\varphi_{t+T^*}(\eta_v, z_v) = \varphi_t(\eta_v, z_v)$ with the initial condition (η_v, z_v). The flow has a fixed point $\varphi_{T^*}(\eta^*, z^*) = (\eta^*, z^*)$. Assume that $\mathscr{O}$ is a periodic orbit that is obtained from the periodic flow

$$\mathscr{O} = \{\varphi_t(\eta^*, z^*) \in \bigcup_{v\in\mathbb{V}} \Phi_v(\mathbb{D}_v) | 0 \le t \le T^*\}. \quad (41)$$

Similarly, we can define a periodic orbit in the partial hybrid zero dynamics as

$$\mathscr{O}_z = \{\varphi^z_t(y_1^*, 0, z^*) \in \bigcup_{v\in\mathbb{V}} \Phi^z_v(\mathbb{D}_v) | 0 \le t \le T^*\}, \quad (42)$$

where $\varphi^z_t(y_1^*, 0, z^*)$ is the flow of the partial zero dynamics with the initial point $(y_1^*, 0, z^*)$. Given the periodic orbit in the partial zero dynamics $\mathscr{O}_z$, we can reconstruct the periodic orbit of the full order dynamics, by using the canonical embedding, $\iota_0(y_{1,v}, z_v) = (y_{1,v}, 0, z_v)$. We can define the norm $\|(\eta_v, z_v)\|_{\mathscr{O}}$ as the closest distance to the $\mathscr{O}$. The periodic orbit $\mathscr{O}$ is exponentially stable if there are constants $r, \delta_1, \delta_2 > 0$ such that if $(\eta_v, z_v) \in \mathbb{B}_r(\mathscr{O})$ it follows that $\|\varphi_t(\eta_v, z_v)\|_{\mathscr{O}} \le \delta_1 e^{-\delta_2 t} \|(\eta_v, z_v)\|_{\mathscr{O}}$. Exponential stability of $\mathscr{O}_z$ can also be similarly defined.

Theorem 2 in [4] says that for small enough $\varepsilon > 0$, the RES-CLF renders the full order periodic orbit $\mathscr{O}$ exponentially stable. Since $\mathbf{K}_{\varepsilon,\bar{\varepsilon}} \subset \mathbf{K}_\varepsilon$, this naturally extends to controllers of the form (26). Therefore, we need to show that $\mathbf{K}_{\varepsilon,\bar{\varepsilon}}$ is indeed an input to state stabilizing controller for the hybrid control system (27). We have the following theorem, which is the extension of Lemma 6 to affine hybrid systems.

Theorem 1: Let $\mathscr{O}_z$ be an exponentially stable periodic orbit for the hybrid zero dynamics $\mathscr{H}|_z$ transverse to $\mathbb{S}|_z$. Existence of a Lipschitz continuous RES-CLF, $u \in \mathbf{K}_\varepsilon$ (24), that exponentially stabilizes $\mathscr{O} = \iota_0(\mathscr{O}_z)$, implies the existence of a Lipschitz continuous Re-ISS-CLF, $u \in \mathbf{K}_{\varepsilon,\bar{\varepsilon}}$ (26), that exponentially input to state stabilizes $\mathscr{O}$.

Proof: [Sketch] A sketch of the proof is provided here due to space constraints. We will use most of the concepts from [4]. Zero stability is valid by default. Therefore, we just need to show that the states are ultimately bounded by a class $\mathscr{K}$ function of $\|d\|_\infty$. for $\|(\eta, z)\|_{\mathscr{O}} \ge \iota(\|d\|_\infty)$, we know that with sufficiently small $\bar{\varepsilon}$ in (26), we can retain the original convergence rate as indicated by (20). Therefore, for the continuous dynamics, $\dot{V}_\varepsilon \le -\frac{\gamma}{\varepsilon} V_\varepsilon$ for $\|(\eta, z)\|$ sufficiently large. With this inequality, all of the formulations from equations (61) to (67) in [4] can be used. In other words, the periodic orbit $\mathscr{O}$ is exponentially converging for sufficiently

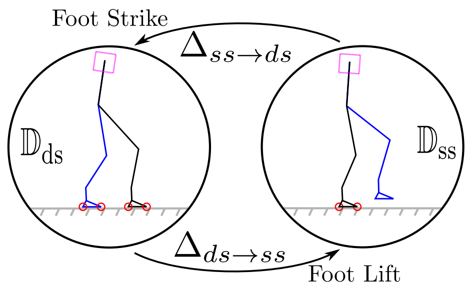

Fig. 4: Hybrid system model for the walking robot DURUS.

large $\|(\eta,z)\|$, meaning the periodic orbit $\mathscr{O}$ is exponential input to state stable. ∎

VI. Results

For verification of the improved stabilizing results presented above, we simulate a humanoid robot under various disturbances and observe improvements of the stability of the gait. The robot under consideration is DURUS, a 23 DOF robot, consisting of fifteen actuated joints and one linear passive spring at the end of each leg. The generalized coordinates of the robot are described in Fig. 1 (see [9]) and the continuous dynamics of the bipedal robot is given by (30). The nominal walking gait considered in this simulation study has two phases: single support, and double support, as shown in Fig. 4. A stable reference walking gait is obtained and verified via an offline optimization algorithm. Therefore, based on Theorem 2 of [4], there is a small enough ε (observed to be ≤ 0.2) that makes the hybrid periodic orbit exponentially stable. It is important to note that the torque requirements increase with the decrease in ε.

The main objective of performing a perturbation analysis is to test the stability of the walking gait under uncertainties that are as realistic as possible. Therefore, we set torque limits of 250Nm for each joint and apply a modeling error of 10% to the mass-inertial properies of the robot. Specifically the modeling error was enforced on the mass, center of mass and inertial properties of each link. It is assumed that other properties such as links lengths and spring constants are accurate. The stabilizing controller chosen for simulation is IO linearization (as given by (34))

$$u_{\text{IO}} = \begin{bmatrix} L_g y_{1,v} \\ L_g L_f y_{2,v} \end{bmatrix}^{-1} \left(- \begin{bmatrix} L_f y_{1,v} \\ L_f^2 y_{2,v} \end{bmatrix} + \begin{bmatrix} -\frac{1}{\varepsilon} y_{1,v} \\ -\frac{2}{\varepsilon} L_f y_{2,v} - \frac{1}{\varepsilon^2} y_{2,v} \end{bmatrix} \right).$$

The ISS controller chosen was (as given by (17))

$$u_{\text{ISS}} = u_{\text{IO}} - \frac{1}{\bar{\varepsilon}} L_g V^T.$$

where the Lyapunov function $V(\eta_v) = \eta_v^T P_\varepsilon \eta_v$. P_ε, depends on ε, is the solution to the CARE: $F^T P_\varepsilon + P_\varepsilon F - \frac{1}{\varepsilon} P_\varepsilon G G^T P_\varepsilon + \frac{1}{\varepsilon} Q_\varepsilon$ (see equation (47) of [4]).

Two test cases were considered: lateral push force to the hip for a duration of 0.1s at the beginning of the single support domain, and stepping onto an unknown ground height. Table I shows the comparison for the push force recovery between u_{IO} and u_{ISS} for different values of $\varepsilon, \bar{\varepsilon}$. It can be observed that with u_{ISS} the robot can handle greater push forces. With lower ε, the stability of the robot is affected (due to 10% model error and torque saturations) resulting in poorer performance for $\varepsilon = 0.05$. On the other hand, Fig. 5 shows that the convergence improves as $\bar{\varepsilon}$ is lowered. Fig. 6 shows the Lyapunov function comparisons for the push recovery. Fig. 7 and Fig. 8 show the comparisons for unknown step over different heights. Fig. 9 shows tiles of push recovery (top) and stepping over (bottom) for an ISS controller. A video link demonstrating the simulations performed on the robot is given in [1].

Controller	IO Gain (ε)	Maximum Allowable Push (N)
IO	0.2	380
	0.1	420
	0.05	395
ISS ($\bar{\varepsilon} = 0.1$)	0.2	380
	0.1	435
	0.05	410
ISS ($\bar{\varepsilon} = 0.01$)	0.2	435
	0.1	435
	0.05	405

TABLE I: Comparison of maximum recoverable push forces in lateral direction. The ISS based controller can handle greater pushes. Also reducing ε leads to instability due to the constraints on model uncertainty and torque limits.

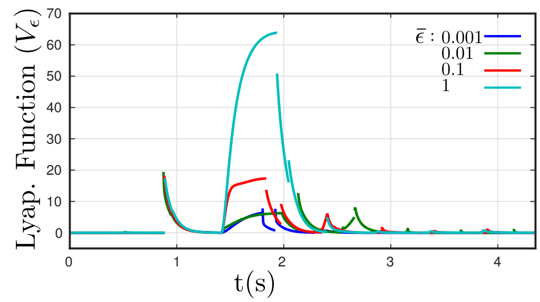

Fig. 5: Comparisons of the Lyapunov function for various values of $\bar{\varepsilon}$ for push recovery. The push force was 350N. The convergence is quicker for decreasing $\bar{\varepsilon}$. The jumps are due to discrete events (impacts).

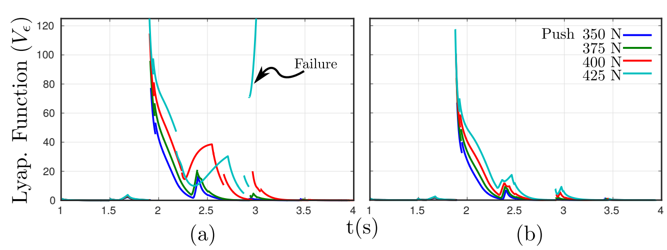

Fig. 6: Push recovery comparison via the Lyapunov functions for IO (a) and ISS (b) based controllers. $\varepsilon = \bar{\varepsilon} = 0.1$. The convergence rate is preserved for ISS-CLF.

VII. Conclusions

In this work, it was shown how to obtain a class of input to state stabilizing controllers for hybrid systems, given the

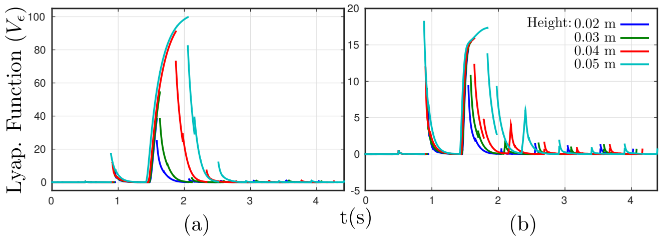

Fig. 7: Step over comparison via Lyapunov functions for IO (a) and ISS (b) based controllers. $\varepsilon = \bar{\varepsilon} = 0.1$. The convergence rate is preserved for ISS-CLF.

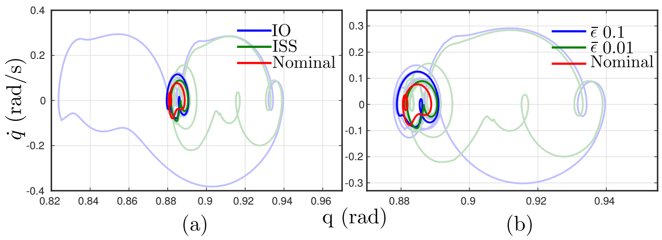

Fig. 8: Walking over 5cm step height. Phase portraits for vertical z position of the torso base are shown here. The ISS based controller shows a much smaller deviation.

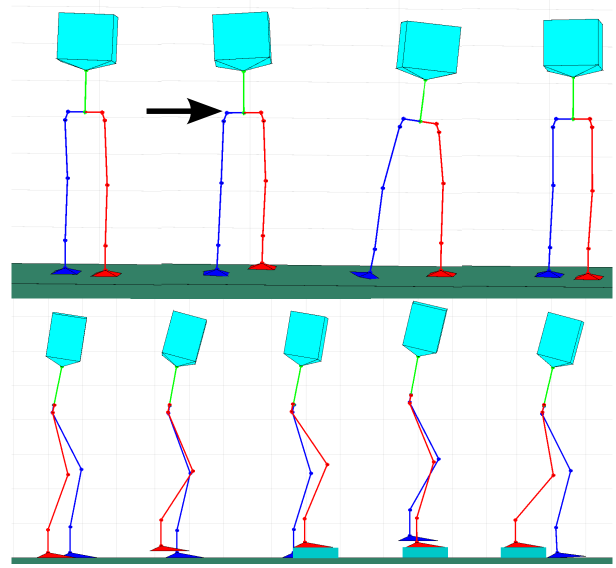

Fig. 9: The top tiles show push recovery and the bottom tiles show stepping onto an unknown disturbance for an ISS controller. Push force of 350N is enforced (second tile) for 0.1s and the reactions are seen in tiles 3 and 4.

set of stabilizing controllers. It was shown in the specific case of the biped robot DURUS by picking a Lyapunov function based on IO linearization. With this construction, we obtained the class of input to state stabilizing controllers (26) that adds robustness to the given hybrid periodic orbits $\mathscr{O}$. The simulation results demonstrated that the auxiliary gain $\bar{\varepsilon}$ can be used to restrict the ultimate bound of the outputs without compromising on the convergence rate $\frac{\gamma}{\varepsilon}$ provided by the RES-CLF (24). Note that the rate of the stabilizing controller ε does not promise the original convergence rate under uncertainties. The methodology shown can be used to realize robust quadratic programs in real time with the end result being input to state stable walking on DURUS.

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